# PINNING AND BINDING ENERGIES FOR VORTICES IN NEUTRON STARS: COMMENTS ON RECENT RESULTS

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We investigate when the energy that pins a superfluid vortex to the lattice of nuclei in the inner crust of neutron stars can be approximated by the energy that binds the vortex to a single nucleus. Indeed, although the pinning energy is the quantity relevant to the theory of pulsar glitches, so far full quantum calculations have been possible only for the binding energy. Physically, the presence of nearby nuclei can be neglected if the lattice is dilute, namely with nuclei sufficiently distant from each other. We find that the dilute limit is reached only for quite large Wigner-Seitz cells, with radii $R_{\mathrm{WS}} \gtrsim 55$ fm; these are found only in the outermost low-density regions of the inner crust. We conclude that present quantum calculations do not correspond to the pinning energies in almost the entire inner crust and thus their results are not predictive for the theory of glitches.

*Keywords*: neutron stars, pulsar glitches, vortex pinning, superfluid neutron matter

## 1. Pinning and binding energies

Pulsar glitches are sudden spin-ups in the otherwise steadily decreasing frequency of rotating magnetized neutron stars. According to the vortex-model, glitches may represent direct evidence for the existence of a macroscopic superfluid inside such stars.[1] This scenario involves the inner crust of the star, namely the density range $\rho_d \leq \rho \leq 0.6\rho_o$, where $\rho_d = 4 \times 10^{11}$ g/cm$^3$ is the neutron drip density and $\rho_o = 2.8 \times 10^{14}$ g/cm$^3$ is the nuclear saturation density. Here, the quantized vortices that form in the superfluid of unbound neutrons to carry its angular momentum can attach themselves to the lattice of neutron-rich nuclei co-existing with the neutrons in the inner crust, thus freezing part of the superfluid angular momentum. As a consequence of the star's spin-down, hydrodynamical forces then develop (Magnus force) which tend to detach the vortices from the lattice in order to let the angular momentum free to decrease. If a large number of vortices can be unpinned simultaneously and deliver their angular momentum to the star surface, its corresponding sudden spin-up is observed as a glitch. The process naturally repeats itself so that part of the superfluid angular momentum is released in discrete time steps, thus explaining the approximate periodicity of glitches.

A crucial microscopic input of the model is the structure of the nuclear lattice at

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the different densities found in the inner crust. Calculations predict a bcc structure for the coulomb lattice, namely each nucleus is at the center of a cubic cell of side $\ell_p = 2R_{\mathrm{WS}}$ with nuclei at each vertex. The bcc lattice can also be seen as a superposition of layers of cubic cells; each layer, of thickness $\ell_p$, is delimited by two planes of nuclei (those at the vertices of the cells) and contains in its middle a third plane of nuclei (those at the center of the cells). In Table 1 we list the results from the classical paper by Negele and Vautherin[2] for five zones along the inner crust. The parameters which characterize the lattice structure are the baryon mass density $(\rho_{\mathrm{B}})$, the number density $(n_{\mathrm{G}})$ and corresponding Fermi momentum $(k_{\mathrm{F,G}})$ of the gas of unbound neutrons, the radius of the Wigner-Seitz (WS) cell $(R_{\mathrm{WS}})$ and the radius of the nucleus $(R_{\mathrm{N}})$ at the center of the cell.

Table 1.   Properties of five zones along the inner crust, as obtained in Ref. 2.

| Zone | $\rho_{\mathrm{B}}$ (g/cm$^3$) | $n_{\mathrm{G}}$ (fm$^{-3}$) | $k_{\mathrm{F,G}}$ (fm$^{-1}$) | $R_{\mathrm{WS}}$ (fm) | $R_{\mathrm{N}}$ (fm) |
|---|---|---|---|---|---|
| 1 | $1.5 \times 10^{12}$ | $4.8 \times 10^{-4}$ | 0.24 | 44.0 | 6.0 |
| 2 | $9.6 \times 10^{12}$ | $4.7 \times 10^{-3}$ | 0.52 | 35.5 | 6.7 |
| 3 | $3.4 \times 10^{13}$ | $1.8 \times 10^{-2}$ | 0.81 | 27.0 | 7.3 |
| 4 | $7.8 \times 10^{13}$ | $4.4 \times 10^{-2}$ | 1.09 | 19.4 | 6.7 |
| 5 | $1.3 \times 10^{14}$ | $7.4 \times 10^{-2}$ | 1.30 | 13.8 | 5.2 |

The other crucial microscopic input is the vortex-lattice interaction potential as a function of the vortex position in the lattice, from which the pinning force could be derived. Due to the complicated spatial geometry of the general problem, however, a simpler approach has been followed so far which calculates only the *pinning energy*. This is defined as the difference in energy between two vortex-lattice configurations which are relevant to the pinning mechanism and possess a reasonable degree of symmetry. More precisely, one starts by taking the vortex axis oriented along a symmetry direction of the lattice and then considering only the interaction of the vortex with the nuclei lying in a plane perpendicular to the vortex axis (the middle plane of a layer). This allows to determine the pinning energy 'per site' (i.e. per layer). The pinning energy 'per unit length', which is the one relevant to the vortex-model for glitches, can then be obtained if one knows the number of pinning sites (layers) per unit length. Statistical estimates of this quantity allow to evaluate pinning also for random vortex-lattice mutual directions.

In Figure 1 (left) the black disks represent the nuclei in the given plane, each positioned at the center of a cube. The nuclei at the vertices of the cubes are shown with white disks; they lie in the planes that delimit the layer, at a distance $R_{\mathrm{WS}}$ above and below the original plane. The vortex core position is schematically represented by the dashed line and the vortex axis, perpendicular to the plane, is shown by a star. The nuclear pinning (NP) configuration corresponds to the vortex pinned on a nucleus, e.g. in position (1). The interstitial pinning (IP) configuration
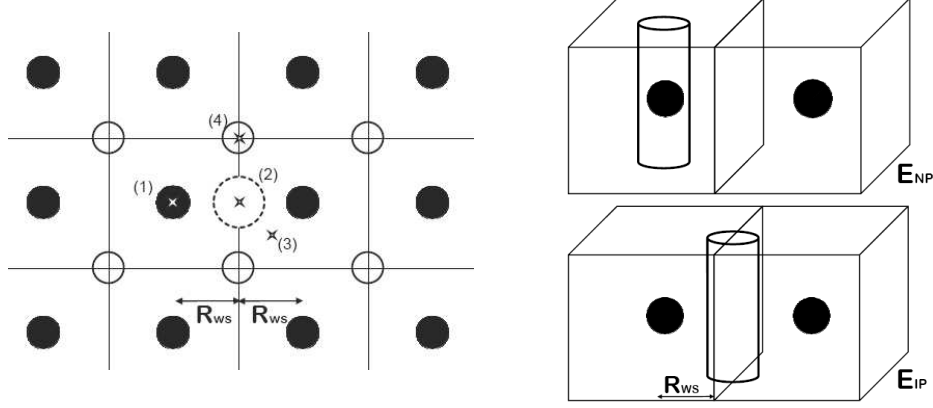
Fig. 1.    Left: vortex positions in a bcc lattice.  Right: configurations for the pinning energy.

corresponds to the vortex away from and equidistant between two nearby nuclei in the plane, e.g. in position (2). Notice that in the NP configuration the vortex also pins to the nuclei in the layers above and below, which are aligned along its axis with spacing $2R_{\mathrm{WS}}$. Conversely, in the IP configuration the vortex does not pin to any nucleus in the lattice. Position (4) is not allowed as an interstitial pinning site; actually, it is physically equivalent to nuclear pinning, since the vortex will pin to the nuclei in the layers above and below; in other words, positions (1) and (4) are equivalent once we consider the pinning energy per unit length.

The pinning energy is the energy necessary to move the vortex by a distance $R_{\mathrm{WS}}$, from the NP to the IP configuration. The *pinning energy per site* is then defined as $E_{\mathrm{pin}} = E_{\mathrm{NP}} - E_{\mathrm{IP}}$, i.e. the difference in energy between the two configurations. Physically, the vortex interacts significantly only with first neighbors; therefore, it is enough to restrict the attention to a system of two adjacent cells, as shown in Figure 1 (right), since contributions from cells further away will cancel out. If $E_{\mathrm{pin}} < 0$ the vortex is attracted by the nucleus ('nuclear pinning') and the average pinning force per site is obtained as $F_{\mathrm{pin}} = |E_{\mathrm{pin}}|/R_{\mathrm{WS}}$. If $E_{\mathrm{pin}} > 0$ the vortex is repelled by the nucleus ('interstitial pinning'); is such a case the pinning force is several orders of magnitude smaller than $F_{\mathrm{pin}}$ and practically negligible. This because the vortex can be moved around the lattice completely avoiding the nuclei, e.g. from (2) to (3), which cost almost no energy.[3]

If one considers a *single* nucleus instead of the actual lattice of nuclei, a simpler quantity can be introduced: the *binding energy* of the vortex-nucleus system is the energy necessary to detach the vortex from the nucleus and take it to infinity, where the background matter is uniform (unaffected by the presence of the nucleus or the vortex). It can be defined as the difference in energy between two configurations, one with the vortex bound to the nucleus (i.e. with vortex-nucleus separation $d = 0$) and the other with the vortex far away and unbound from the nucleus (i.e. with vortex-nucleus separation $d \to \infty$). The short range nature of nuclear forces and

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the rapidly decreasing kinetic potential ($\propto 1/r^2$) associated to the vortex flow, however, imply that the vortex and the nucleus interact significantly only when they are relatively close to each other. Therefore, to a good approximation the binding energy is reached not for $d \to \infty$, but already for some finite separation $d = D_{\mathrm{conv}}$. The convergence distance $D_{\mathrm{conv}}$ corresponds to a configuration presenting a zone between the nucleus and the vortex where neutron matter has practically reached uniformity; taking $d > D_{\mathrm{conv}}$ only adds more uniform matter in between, whose contributions cancel out in the binding energy.

In Figure 2 (left) the vortex and the nucleus are shown at a distance $d$ apart. Such a system can be described by two cylinders of radius $R = \frac{1}{2}d$ and height $h$, which are also natural quantization boxes to evaluate the energy of this geometry. Indeed, the vortex-nucleus bound (B) and unbound (U) configurations are those represented in Figure 2 (right) and the binding energy is defined as $E_{\mathrm{bind}} = E_{\mathrm{B}} - E_{\mathrm{U}}$. Of course, for this to make sense the result must converge; as explained, this will happen only when the separation is $d \geq D_{\mathrm{conv}}$ and thence when the cylinders have radius larger than $R_{\mathrm{conv}} = \frac{1}{2}D_{\mathrm{conv}}$. The physical interpretation of the convergence radius is the same as before: when $R = R_{\mathrm{conv}}$, matter has practically reached uniformity at the surface of both cylinders of Figure 2. Taking larger boxes would only add more uniform matter at their boundaries, whose energy contributions cancel out in the binding energy. In this respect, we notice that the choice of cylindrical boxes, although natural, is by no means unique; for example one could as well take two cubic boxes of side $\ell = d$. As long as $\ell \geq 2R_{\mathrm{conv}}$, energy contributions from the additional uniform matter will again cancel out in the final binding energy.

The complicated geometry of the pinning configurations, with its interplay of different symmetries (spherical for the nucleus, axial for the vortex, periodic for the lattice), makes the calculation of pinning energies a very difficult problem at the quantum level. Conversely, binding energies are simpler to calculate since the
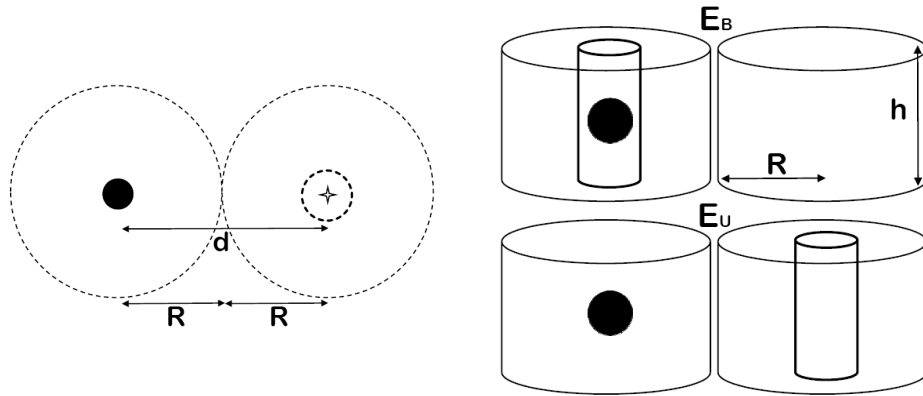


Fig. 2.   Left: vortex-nucleus system with separation $d$ and cylindrical quantization boxes of radius $R = \frac{1}{2}d$.   Right: configurations for the binding energy.

convergence requirement allows to split both the U and B configurations into independent axially symmetric problems (the pairs of cylinders). The question is under which conditions pinning and binding energies can be expected to be equivalent.

## 2. Dilute limit for the lattice

In general, pinning and binding energies are different and unrelated quantities. The binding energy represents the interaction between a vortex and a single nucleus while pinning involves the interaction of the vortex with a lattice of nuclei. Moreover, binding is defined as the value to which the difference $E_{\mathrm{B}} - E_{\mathrm{U}}$ converges for large enough vortex-nucleus separation and thus it does not depend on any distance parameter; conversely, pinning is defined with respect to a specific configuration, namely the difference $E_{\mathrm{NP}} - E_{\mathrm{IP}}$ must be evaluated at vortex-nucleus separation $R_{\mathrm{WS}}$, and thus it depends crucially on the length parameter of the lattice. Figure 3 (left) illustrates the difference between $E_{\mathrm{bind}}$ and $E_{\mathrm{pin}}$: binding is related only to the value of the interaction potential at the center of the WS cells, while pinning is determined by the local extremum in the interaction potential which, by symmetry reasons, must develop at the boundary between two WS cells. Whether this is a maximum or a minimum will depend on the particular radial dependence of the interaction potential; as long as this dependence is not known, as it is the case for the problem under study, no prediction on the value or *even the sign* of the pinning energy can be extracted from the knowledge of the binding energy, as obvious from the figure. We point out that the NP configuration of pinning is physically equivalent to the B configuration of binding, at least as long as the vortex core is smaller than the WS radius so that the presence of nearby nuclei can be neglected. Therefore, the difference between $E_{\mathrm{pin}}$ and $E_{\mathrm{bind}}$ follows essentially from the physical difference between the IP and U configurations.

If the spacing of the nuclei in the lattice is sufficiently large, however, one expects that in the IP configuration the vortex is sufficiently distant from the nuclei that it does not interact significantly with them anymore, so that the IP configuration
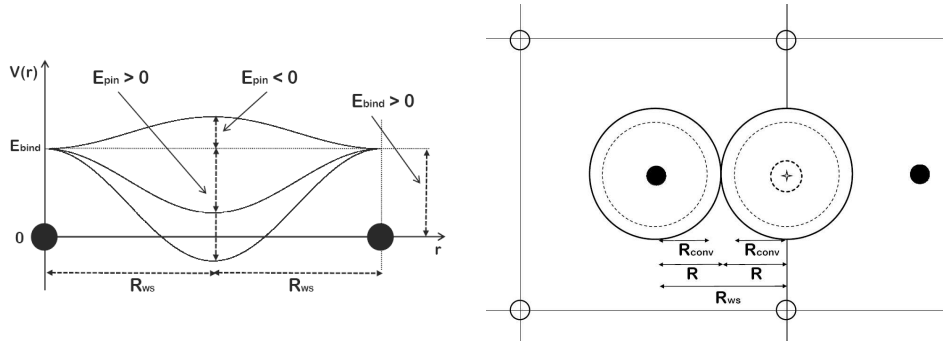


Fig. 3.   Left: example of different possible pinning energies corresponding to the same binding energy.   Right: lattice in the dilute limit $R_{\mathrm{WS}} > 2R_{\mathrm{conv}}$.

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becomes physically equivalent to the U configuration. If this is the case, the pinning energy will coincide with the binding energy; we call this scenario, where $E_{\text{pin}} = E_{\text{bind}}$, the *dilute limit* for the lattice. Quantitatively, such a limit is reached when the vortex-nucleus distance in the IP configuration ($R_{\text{WS}}$) is larger than the separation for which they do not appreciably interact anymore ($D_{\text{conv}}$).

Figure 3 (right) shows the lattice in the dilute limit, namely with $R_{\text{WS}} > D_{\text{conv}} = 2R_{\text{conv}}$. The IP configuration with vortex-nucleus separation $d = R_{\text{WS}}$ can be represented by two cylinders (solid circles) of radius $R = \frac{1}{2}d = \frac{1}{2}R_{\text{WS}} > R_{\text{conv}}$. Since this radius is larger than the convergence radius (dotted circles), the IP configuration in this limit is indeed equivalent to the U configuration (cf. Figure 2) and the pinning energy thus calculated is the same as the binding energy. Conversely, if $R_{\text{WS}} < D_{\text{conv}} = 2R_{\text{conv}}$ the lattice is in the dense limit and $E_{\text{pin}} \neq E_{\text{bind}}$. Figure 4 (left) illustrates this scenario; the cylinders needed to represent the IP configuration have radii smaller than $R_{\text{conv}}$ and therefore this geometry is not equivalent to the U configuration, which requires convergence to be properly defined. Consequently, the energy difference $E_{\text{pin}}$ calculated from this geometry cannot be equal to $E_{\text{bind}}$, the latter being reached only for larger radii of the cylinders.

In conclusion, the important parameter to be determined is the convergence distance $D_{\text{conv}}$ for binding. A lower limit can easily be found by considering the kinetic energy contribution of a nucleus added to the vortex flow at distance $d$ from the vortex axis.[4] This energy becomes negligible only for vortex-nucleus separations larger than $d \approx 30$ fm,[5] which implies that $D_{\text{conv}}$ cannot be smaller than this value.

## 3. Comments on recent results

To date, the only consistent and realistic calculation of pinning energies corresponding to the parameters of the inner crust (Table 1) have been performed in the framework of the Local Density Approximation (LDA); details of the model are given in Ref. 5. The purely additive nature of energy contributions from adjacent volumes of
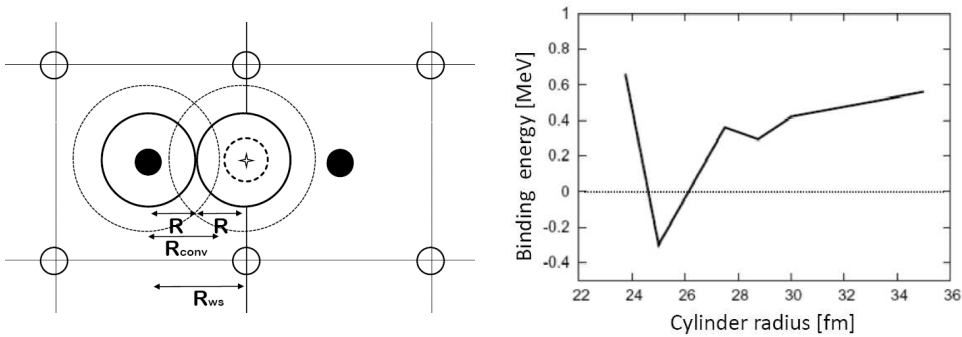


Fig. 4.   Left: lattice in the dense limit $R_{\text{WS}} < 2R_{\text{conv}}$.  Right: dependence of the binding energy on the radius of the quantization cylinder (figure taken from Ref. 7).

matter in this semi-classical approximation allows on the one hand to deal with the difficult geometry of pinning; but on the other hand, it completely neglects 'proximity' effects associated to the non-local quantum nature of pairing phenomena. In the absence of credible quantum calculations for the vortex-lattice system, however, which can confirm or modify the semi-classical predictions, the LDA results are the only physically reasonable input for the vortex theory of glitches.

Binding energies are relevant to glitch theory in two related aspects: they allow to determine the convergence distance $D_{\mathrm{conv}} = 2R_{\mathrm{conv}}$ and they are equal to the pinning energies for the density regions where the lattice is dilute, i.e. where $R_{\mathrm{WS}} > D_{\mathrm{conv}}$. Recently, a quantum calculation of binding energies has been performed by solving the mean-field Hartree-Fock-Bogoliubov (HFB) equations. Although improperly and quite misleadingly called pinning energy, the quantity calculated in Ref. 6 is obtained precisely as shown in Figure 2 (right), and convergence is correctly required as a consistency condition for the model. In Figure 4 (right) we show their study of the convergence of $E_{\mathrm{bind}}$ as a function of the cylinder radius $R$. At the value $R = 30$ fm used in Ref. 6, *reasonable* convergence is finally reached (the still increasing $E_{\mathrm{bind}}$ is included in the error bars in their figures). Being optimistic one may go down to $R_{\mathrm{conv}} \approx 27 - 28$ fm, but lower values present wild oscillations (at $R \sim 25$ fm, $E_{\mathrm{bind}}$ even reverses its sign), obviously out of control and not related to any physics but only to the artificial boxes of the quantization procedure (the 'dripped' neutrons occupy positive-energy *continuum* states). One thus finds $D_{\mathrm{conv}} \approx 55$ fm, and thence the dilute limit corresponds to $R_{\mathrm{WS}} \gtrsim 55$ fm. The results of Ref. 2 show that this condition corresponds to only the outermost layers of the inner crust, with densities $\rho_{\mathrm{B}} \lesssim 4.7 \times 10^{11}$ g/cm$^3$ and $n_{\mathrm{G}} \lesssim 10^{-4}$ fm$^{-3}$ ($k_{\mathrm{F,G}} \lesssim 0.14$ fm$^{-1}$). This is very unwelcome, since it does not allow to use binding in place of pinning in most of the crust, particularly in the density regions which appears to be relevant to glitches in the LDA, i.e. around zone 3.[5]

In Figure 5 (left) we report the comparison between LDA and HFB results used in Ref. 6 to support their main conclusion, namely that around zones 3 and 4 pinning is not nuclear, but interstitial. This conclusion is wrong and misleading: as shown by the vertical line at $k_{\mathrm{F,G}} = 0.14$ fm$^{-1}$, the figure compares two different energies in a regime where they do not represent the same quantity. To state it more vividly, in order to describe pinning in zone 1 ($R_{\mathrm{WS}} = 44$ fm) the HFB approach should use cylinders with $R = 22$ fm and the result should already converge at this small radius; to describe zone 3, convergence should be obtained at $R = 14$ fm!

Where the dilute limit is reached, however, the HFB results for $E_{\mathrm{bind}}$ can and should be compared to the LDA results for $E_{\mathrm{pin}}$, in order to assess whether the semi-classical model is realistic or not, at least in this regime. One can actually exploit the 'locality' of the LDA to extend the analysis a bit further. Indeed, the absence of proximity effects is such that, when applied to the calculation of $E_{\mathrm{bind}}$, the LDA converges to a definite final value already for $D_{\mathrm{conv}} \approx 30$ fm. Therefore, at densities $n_{\mathrm{G}} \lesssim 0.01$ fm$^{-3}$ ($k_{\mathrm{F,G}} \lesssim 0.6$ fm$^{-1}$), where $R_{\mathrm{WS}} \gtrsim 30$ fm, the LDA values for pinning are the same as its results for binding and as such they could be
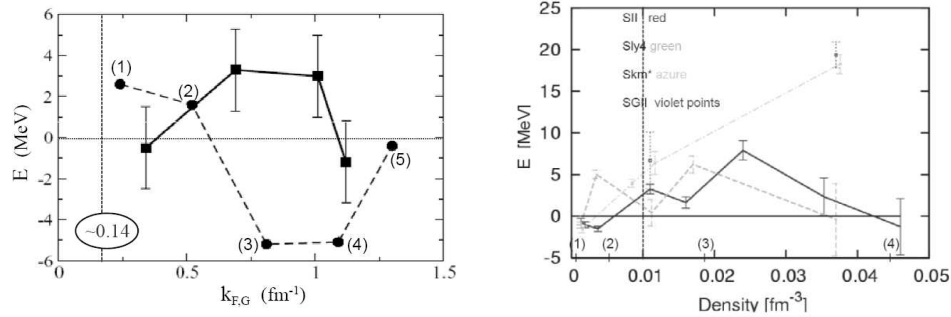
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Fig. 5.   Left: comparison of LDA results for pinning energies (dashed line) and HFB results for binding energies (solid line); the numbers indicate the zones of Table 1 (figure taken from Ref. 6). Right: comparison of HFB results for binding energies corresponding to different choices of the pairing interaction (figure taken from Ref. 7).

compared to the corresponding HFB results. However, before claiming that at very low densities (between zones 1 and 2) the HFB and LDA predictions for binding have opposite sign, like Figure 5 (left) may suggest, some crucial issues should be kept in mind: *(i)* Figure 5 (right) shows the HFB results for different pairing interactions; the corresponding binding energies between zones 1 and 2 can differ even in the sign. Thence the comparison must be done with the 'same' pairing interaction, in the sense that the density dependence of the neutron gap in uniform matter must be the same in HFB and LDA. Instead, the interaction used in Ref. 6 is equivalent to Gogny D1S, which for $k_{F,G} > 1$ fm$^{-1}$ yields larger gaps than the Argonne used in the LDA. *(ii)* As carefully explained in Ref. 5, the pinning energy should be evaluated at *fixed chemical potential* (the particle bath represented by the macroscopic inner crust which surrounds the widely spaced vortices), while the results of Ref. 6 correspond to a fixed number of particles. Since $E_{\text{bind}}$ comes from the difference between very large numbers, great attention must be devoted to this kind of issue. *(iii)* Different zones present nuclei with different radii, while in Ref. 6 the same nucleus (with $R_N = 7$ fm) was taken at all densities.

    With these issues under control, the comparison of quantum and semi-classical results for binding at low densities could become quite instructive.

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